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System of Generalized Vector Quasi-Equilibrium Problems with Applications to Fixed Point Theorems for a Family of Nonexpansive Multivalued Mappings

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Abstract. In this paper, we establish the existence theorems of the generalized vector quasi-equilibrium problems. From some existence theorem, we establish fixed point theorems for a family of lower semicontinuous or nonexpansive multivalued mappings. We also obtain the existence theorems of system of mixed generalized vector variational-like inequalities and existence theorems of the Debreu vector equilibrium problems and the Nash vector equilibrium problems.

Key words: generalized vector quasi-equilibrium problem, nonexpansive multivalued map, quasiconvex, quasiconvex-like, upper(lower) semicontinuous multivalued map

1. Introduction

Let E be a topological vector space (in short t.v.s.), X be a nonempty subset of E and $f : X \times X \to R$ be a function with $f(x, x) \geqslant 0$ for all $x \in X$, then the scalar equilibrium problem is to find $\bar{x} \in X$ such that

$$f(\bar{x}, y) \geqslant 0 \quad \text{for all} \quad y \in X.$$

The equilibrium problem contains optimization problems, variational inequalities problems, the Nash equilibrium problems, fixed point problems and complementary problems as special cases (see [5]). This problem was extensively investigated and generalized to the vector equilibrium problems for single valued or multivalued mappings [8,12,15,19–24].

Let I be an index set. For each $i \in I$, let Z_i be a Hausdorff t.v.s. and let X_i and D_i be nonempty subsets of two Hausdorff t.v.s. E_i and V_i, respectively. Let $S_i : X \multimap X_i$, $T_i : X \multimap D_i$, $C_i : X \multimap Z_i$ be multivalued mappings with nonempty values and let $F_i : D_i \times X \times X_i \multimap Z_i$ be a multivalued mapping, where $X = \Pi_{i \in I} X_i$.

In this paper, we study the following classes of the system of the generalized vector quasi-equilibrium problems:

(1) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I$, $\bar{x}_i\in clS_i(\bar{x})$, $F_i(t_i,\bar{x},y_i)\subset C_i(\bar{x})$ for all $y_i\in S_i(\bar{x})$, and for all $t_i\in T_i(\bar{x})$;
(2) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I$, $\bar{x}_i\in clS_i(\bar{x})$ and for each $y_i\in S_i(\bar{x})$, there exists $t_i\in T_i(\bar{x})$ such that $F_i(t_i,\bar{x},y_i)\cap C_i(\bar{x})\neq\emptyset$;
(3) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for each $i\in I$, $\bar{x}_i\in clS_i(\bar{x})$, $F_i(t_i,\bar{x},y_i)\cap(-\text{int}C_i(\bar{x}))=\emptyset$ for all $y_i\in S_i(\bar{x})$ and all $t_i\in T_i(\bar{x})$;
(4) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for each $i\in I$, $\bar{x}_i\in \text{cl}S_i(\bar{x})$, and for each $y_i\in S_i(\bar{x})$, there exists $t_i\in T_i(\bar{x})$ such that $F_i(t_i,\bar{x},y_i)\not\subset(-\text{int}C_i(\bar{x}))$.

Recently Lin et al. [26] studied the following problems:

(i) find $(\bar{x},\bar{y})\in X\times Y$ such that for each $i\in I, \bar{x}_i\in clS_i(\bar{x}), \bar{y}_i\in clT_i(\bar{x})$ and $f_i(\bar{x},\bar{y},u_i)\subset C_i(\bar{x})$ for all $u_i\in S_i(\bar{x})$;
(ii) find $(\bar{x},\bar{y})\in X\times Y$ such that $\bar{x}_i\in clS_i(\bar{x}), \bar{y}_i\in clT_i(\bar{x})$ and $f_i(\bar{x},\bar{y},u_i)\cap C_i(\bar{x})\neq\emptyset$ for all $u_i\in S_i(\bar{x})$;
(iii) find $(\bar{x},\bar{y})\in X\times Y$ such that for each $i\in I$, $\bar{x}_i\in clS_i(\bar{x}), \bar{y}_i\in clT_i(\bar{x})$ and $f_i(\bar{x},\bar{y},u_i)\cap(-\text{int}C_i(\bar{x}))=\emptyset$ for all $u_i\in S_i(\bar{x})$;
(iv) find $(\bar{x},\bar{y})\in X\times Y$ such that for each $i\in I, \bar{x}_i\in clS_i(\bar{x}), \bar{y}_i\in clT_i(\bar{x})$ and $f_i(\bar{x},\bar{y},u_i)\not\subset(-\text{int}C_i(\bar{x}))$ for all $u_i\in S_i(\bar{x})$,

where X_i and Y are nonempty subsets of two Hausdorff locally convex t.v.s., Z_i is a real Hausdorff t.v.s., $S_i: X=\Pi_{i\in I}X_i\multimap D_i$, $T_i: X\multimap H_i$, $C_i: X\multimap Z_i$, and $f_i: X\times Y\times X_i\multimap Z_i$ are multivalued maps, D_i and H_i are two nonempty compact metrizable subsets of X_i and Y_i, respectively.

Lin et al. [25] also studied the following problems:

(i′) find $\hat{x},\hat{y}\in X$ such that for each $i\in I, \hat{x}_i\in\bar{S}_i(\hat{x}), \hat{y}_i\in\bar{T}_i(\hat{x})$ and $f_i(\hat{x},\hat{y},u_i)\subset C_i(\hat{x})$ for all $u_i\in S_i(\hat{x})$;
(ii′) find $\hat{x},\hat{y}\in X$ such that for each $i\in I, \hat{x}_i\in\bar{S}_i(\hat{x}), \hat{y}_i\in\bar{T}_i(\hat{x})$ and $f_i(\hat{x},\hat{y},u_i)\cap C_i(\hat{x})=\emptyset$ for all $u_i\in S_i(\hat{x})$;
(iii′) find $\hat{x},\hat{y}\in X$ such that for each $i\in I, \hat{x}_i\in\bar{S}_i(\hat{x}), \hat{y}_i\in\bar{T}_i(\hat{x})$ and $f_i(\hat{x},\hat{y},u_i)\cap(-\text{int}C_i(\hat{x}))=\emptyset$ for all $u_i\in S_i(\hat{x})$;
(iv′) find $\hat{x},\hat{y}\in X$ such that for each $i\in I, \hat{x}_i\in\bar{S}_i(\hat{x}), \hat{y}_i\in\bar{T}_i(\hat{x})$, and $f_i(\hat{x},\hat{y},u_i)\not\subset(-\text{int}C_i(\hat{x}))$ for all $u_i\in S_i(\hat{x})$,

where $f_i: X\times X\times X_i\multimap Z_i$, $T_i: X\multimap X_i$, $C_i: X\multimap Z_i$ and $S_i: X\multimap X_i$ are multivalued maps and $\hat{y}_i\in\overline{T_i}(\hat{x})$ means that $(\hat{x},\hat{y}_i)\in clGrT_i=$ closure of the graph of T_i.

Our problems, our approaches and results are different from Lin et al. [25, 26]. In Lin et al. [25, 26], $T_i(x)$ is assumed to be convex for all $x\in X$, but in this paper, we don't assume that $T_i(x)$ is convex for any $x\in X$.

Problems (1)–(4) contain the problems (1′)–(4′), respectively, as special cases:

(1′) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I, \bar{x}_i\in clS_i(\bar{x})$, $\langle\theta_i(\bar{x}_i,t_i),\eta_i(y_i,\bar{x}_i)\rangle+f_i(\bar{x},y_i)\in C_i(\bar{x})$ for all $y_i\in S_i(\bar{x})$, $t_i\in T_i(\bar{x})$;

(2′) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I, \bar{x}_i\in clS_i(\bar{x})$, and for each $y_i\in S_i(\bar{x})$, there exists $t_i\in T_i(\bar{x})$ such that $\langle\theta_i(\bar{x}_i,t_i),\eta_i(y_i,\bar{x}_i)\rangle+f_i(\bar{x},y_i)\in C_i(\bar{x})$;

(3′) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I, \bar{x}_i\in clS_i(\bar{x})$, and for each $y_i\in S_i(\bar{x})$, there exists $t_i\in T_i(\bar{x})$ such that $\langle\theta_i(\bar{x}_i,t_i),\eta_i(y_i,\bar{x}_i)\rangle+f_i(\bar{x},y_i)\notin(-\text{int}\,C_i(\bar{x}))$, for all $y_i\in S_i(\bar{x})$ and all $t_i\in T_i(\bar{x})$, where $\theta_i:X_i\times D_i\longrightarrow L(E_i,Z_i)$, $\eta_i:X_i\times X_i\longrightarrow E_i$, $f_i:X\times X_i\longrightarrow Z_i$ are functions, $L(E_i,Z_i)=$ the space of continuous linear operators from E_i to Z_i and $D_i\subset L(E_i,Z_i)$;

(4′) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I, \bar{x}_i\in clS_i(\bar{x})$, and for each $y_i\in S_i(\bar{x})$, there exists $t_i\in T_i(\bar{x})$ such that $\langle\theta_i(\bar{x}_i,t_i),\eta_i(y_i,\bar{x}_i)\rangle+f_i(\bar{x},y_i)\notin(-\text{int}\,C_i(\bar{x}))$.

If $\eta_i\equiv 0$, and $f_i(x,y_i)=g_i(x^i,y_i)-g_i(x^i,x_i)$, the above four problems are reduced to the following two types of the Debreu vector equilibrium problems [9].

(1) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I$, $\bar{x}_i\in clS_i(\bar{x})$ and $g_i(\bar{x}^i,y_i)-g_i(\bar{x}^i,\bar{x}_i)\in C_i(\bar{x})$ for all $y_i\in S_i(\bar{x})$;

(2) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that $g_i(\bar{x}^i,y_i)-g_i(\bar{x}^i,\bar{x}_i)\notin(-\text{int}C_i(\bar{x}))$ for all $y_i\in S_i(\bar{x})$.

Recently Ansari et al. [3] studied type (2) the Debreu vector equilibrium problems [9]. If $S_i(x)=X_i$ for all $x\in X$, then these two types of the Debreu vector equilibrium problems will be reduced to the Nash vector equilibrium problems [27]. If $\eta_i\equiv 0$ and $f_i(x,y_i)=g_i(y_i)-g_i(x_i)$, then (1′) and (4′) are reduced to the following problems, respectively:

(1) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I$, $\bar{x}_i\in clS_i(\bar{x})$ and $g_i(y_i)-g_i(\bar{x}_i)\in C_i(\bar{x})$ for all $y_i\in S_i(\bar{x})$;

(2) find $\bar{x}=(\bar{x}_i)_{i\in I}\in X$ such that for all $i\in I$, $\bar{x}_i\in clS_i(\bar{x})$ and $g_i(y_i)-g_i(\bar{x}_i)\notin(-\text{int}C_i(\bar{x}))$ for all $y_i\in S_i(\bar{x})$.

Problems (1)–(4) also contain fixed point problems for any collection of l.s.c. or nonexpansive multivalued maps. Recently, Lin [23] studied problems (1),(3) and (4) under some monotonicity condition when I is a singleton. Fu [12] studied type (3) problem when I is a singleton, $S(x)=X$ for all $x\in X$ and F_i is a single valued function, Khanh et al. [18] studied some special cases of problems (3′) and (4′) with some pseudomonotone assumption when I is a singleton. Till now, we do not know problems (1) and (3) being studied. Recently, Ansari et al. [1] studied system of vector equilibrium problem for single value functions. Ansari et al. [2] studied system of vector quasi-equilibrium problems. Ansari et al. [3] studied the system of generalized vector quasi-equilibrium problems. But our results and methods are quite different from Ref.3. As applications of our result, we establish fixed point theorems for any family of lower semicontinuous or nonexpansive multivalued mappings defined on product of convex subsets of Banach spaces or product of convex subsets of Hilbert spaces. Our results are quite different from the existence results of fixed point theorems (see for examples [13,14,16,17,30]). We give a new approach to the study of fixed point theorems. In the final section, we give some applications to study system of vector mixed variational-like inequality problems from which we can establish the existence theorems of the Debreu vector equilibrium problems and the Nash vector equilibrium problems.

2. Preliminaries

Let X and Y be nonempty sets, a multivalued map $T : X \multimap Y$ is a function from X to the power set of Y. Let $A \subset X$, $B \subset Y$, $x \in X$ and $y \in Y$, we define $T(A) = \bigcup\{T(x) : x \in A\}$; $x \in T^{-}(y)$ if and only if $y \in T(x)$. For topological spaces X and Y, let $T : X \multimap Y$, T is said to be (i) upper semicontinuous (in short u.s.c.) at $x \in X$ if for every open set V in Y with $T(x) \subset V$, there exists an open neighborhood $U(x)$ of x such that $T(x^{'}) \subset V$ for all $x^{'} \in U(x)$; (ii) lower semicontinuous (in short l.s.c.) at $x \in X$ if for every open set V in Y with $T(x) \bigcap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(x^{'}) \bigcap V \neq \emptyset$ for all $x^{'} \in U(x)$; (iii) T is u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; (iv) closed if $GrT = \{(x, y) : x \in X, y \in T(x)\}$ is closed in $X \times Y$.

DEFINITION 2.1 Let X be a convex subset of a t.v.s. and let $C : X \multimap Z$ be a multivalued map such that for each $x \in X$, $C(x)$ is a closed convex cone with nonempty interior. A multivalued function $F : X \times X \multimap Z$ is called

(i) $C(x) - quasiconvex$ if for any $x, y_1, y_2 \in X$ and $\lambda \in [0, 1]$, we have either

$$F(x, y_1) \subset F(x, \lambda y_1 + (1 - \lambda) y_2) + C(x)$$

or

$$F(x, y_2) \subset F(x, \lambda y_1 + (1 - \lambda) y_2) + C(x).$$

(ii) $C(x) - quasiconvex - like$ if for any $x, y_1, y_2 \in X$, $\lambda \in [0, 1]$, we have either

$$F(x, \lambda y_1 + (1 - \lambda) y_2) \subset F(x, y_1) - C(x)$$

or

$$F(x, \lambda y_1 + (1 - \lambda) y_2) \subset F(x, y_2) - C(x).$$

Remark 2.1 If $F: X \times X \to Z$ is a single valued function, then $C(x)-quasiconvex$ and $C(x)-quasiconvex-like$ are equivalent. In this case, we have either

$$F(x, y_1) \in F(x, \lambda y_1 + (1-\lambda) y_2) + C(x)$$

or

$$F(x, y_2) \in F(x, \lambda y_1 + (1-\lambda) y_2) + C(x).$$

Throughout this paper, all topological spaces are assumed to be Hausdorff. The following theorems are need in this paper.

PROPOSITION 2.1 [24]. *Let E_1, E_2 and Z be real t.v.s., X and Y be nonempty subsets of E_1 and E_2, respectively. Let $F: X \times Y \multimap Z$, $S: X \multimap Y$ be multivalued maps.*

(i) *if both S and F are l.s.c., then $T: X \multimap Z$ defined by $T(x) = \bigcup_{y \in S(x)} F(x, y)$ is l.s.c. on X;*
(ii) *if both F and S are u.s.c. with compact values, then T is an u.s.c. multivalued map with compact values.*

PROPOSITION 2.2 [29]. *Let X and Y be topological spaces, $F: X \multimap Y$ be a multivalued map. Then F is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ for every α and y_α converging to y.*

PROPOSITION 2.3 [4]. *Let X and Y be topological spaces, $F: X \multimap Y$ be a multivalued map.*

(i) *if $F: X \multimap Y$ is an u.s.c. multivalued map with closed values, then F is closed;*
(ii) *if F is compact and $F: X \multimap Y$ is an u.s.c. multivalued map with compact values, then $F(X)$ is compact.*

PROPOSITION 2.3 [10]. *Let I be any index set, for all $i \in I$, let X_i be a nonempty convex subset of a t.v.s. E_i and let $S_i: X = \prod_{i \in I} X_i \multimap X_i$ be a multivalued map. Assume that the following conditions hold:*

(i) *for all $i \in I$, and for all $x \in X$, $S_i(x)$ is convex;*
(ii) *for all $i \in I$, $x = (x_i)_{i \in I} \in X$, $x_i \notin S_i(x)$;*
(iii) *for all $i \in I$, and for all $y_i \in X_i$, $S_i^{-}(y_i)$ is open in X; and*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$ there exists $j \in I$ satisfying $S_j(x) \bigcap M_j \neq \emptyset$.*

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

PROPOSITION 2.4 [28]. *Let E be a Banach space with a* Fréchet *differentiable norm. Then the duality mapping $J : E \to E^*$ is norm to norm continuous, where E^* is the dual space of E.*

3. Existence Theorems of System of Generalized Vector Quasi-Equilibrium Problems

Let I be any index set, for each $i \in I$, let E_i, V_i and Z_i be real t.v.s., X_i be a nonempty closed convex subset of E_i, D_i be a nonempty subset of V_i and $X = \prod_{i\in I} X_i$. For each $i \in I$, let $C_i : X \multimap Z_i$ be a multivalued map such that $C_i(x)$ is a proper closed convex cone with apex at the origin and $\text{int}C_i(x) \neq \emptyset$ for each $x \in X$. Let $S_i : X \multimap X_i$, $T_i : X \multimap D_i$ and $F_i : D_i \times X \times X_i \multimap Z_i$ be multivalued maps with nonempty values. Throughout this section, unless otherwise specified, we fixed these notations and assumptions.

THEOREM 3.1 *For each $i \in I$, suppose that*

(i) *for each $x = (x_i)_{i\in I} \in X$ and each $t_i \in D_i$, $F_i(t_i, x, x_i) \subset C_i(x)$;*
(ii) *for each $(t_i, x) \in D_i \times X$, $y_i \multimap F_i(t_i, x, y_i)$ is $C_i(x)-$quasiconvex; that is, for any y_i, $y_i' \in X_i$, $\lambda \in [0, 1]$, either*

$$F_i(t_i, x, y_i) \subseteq F_i(t_i, x, \lambda y_i + (1-\lambda)y_i') + C_i(x)$$

or

$$F_i(t_i, x, y_i') \subseteq F_i(t_i, x, \lambda y_i + (1-\lambda)y_i') + C_i(x);$$

(iii) *$T_i : X \multimap D_i$ and $F_i : D_i \times X \times X_i \multimap Z_i$ are l.s.c. multivalued maps;*
(iv) *$clS_i : X \multimap X_i$ is* u.s.c.*, $S_i^-(y_i)$ is open for all $y_i \in X_i$, $C_i : X \multimap Z_i$ is* u.s.c. *and $S_i(x)$ is a nonempty convex set for each $x \in X$;*
(v) *there exist a nonempty compact subset $K \subset X$ and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exist $j \in I$, $y_j \in M_j \bigcap S_j(x)$ and $t_j \in T_j(x)$ such that $F_j(t_j, x, y_j) \not\subset C_j(x)$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, such that for all $i \in I$, $\bar{x}_i \in clS_i(\bar{x})$ $F_i(t_i, \bar{x}, y_i) \subset C_i(\bar{x})$ for all $y_i \in S_i(\bar{x})$ and all $t_i \in T_i(\bar{x})$.

Proof. Let $A_i = \{x = (x_i)_{i\in I} \in X : x_i \in clS_i(x)\}$. By (iv), it is easy to see that A_i is closed. Let $P_i : X \multimap X_i$ be defined by

$$P_i(x) = \{y_i \in X_i : F_i(t_i, x, y_i) \not\subset C_i(x), \text{for some}\, t_i \in T_i(x)\},$$

then $P_i(x)$ is convex for each $x \in X$. Indeed, if there exist $x_0 \in X$, y_i, $y_i' \in P_i(x_0)$ and $\lambda_0 \in [0, 1]$ such that $\lambda_0 y_i + (1-\lambda_0)y_i' \notin P_i(x_0)$. Then

$F_i(t_i, x_0, \lambda_0 y_i + (1-\lambda_0)y_i^{'}) \subset C_i(x_0)$ for all $t_i \in T_i(x_0)$. By (ii), either

$$F_i(t_i, x_0, y_i) \subset F_i(t_i, x_0, \lambda_0 y_i + (1-\lambda_0)y_i^{'}) + C_i(x_0) \subset C_i(x_0) + C_i(x_0) \subset C_i(x_0)$$

or

$$F_i(t_i, x_0, y_i^{'}) \subset F_i(t_i, x_0, \lambda_0 y_i + (1-\lambda_0)y_i^{'}) + C_i(x_0) \subset C_i(x_0) + C_i(x_0) \subset C_i(x_0)$$

for all $t_i \in T_i(x_0)$.

But $y_i, y_i^{'} \in P_i(x_0)$, there exist $t_i, t_i^{'} \in T_i(x_0)$ such that $F_i(t_i, x_0, y_i) \not\subset C_i(x_0)$ and $F_i(t_i^{'}, x_0, y_i^{'}) \not\subset C_i(x_0)$. This leads to a contradiction. Therefore for each $x \in X$, $\lambda \in [0,1]$ and $y_i, y_i^{'} \in P_i(x)$, there exists $t_i^{''} \in T_i(x)$ such that $F_i(t_i^{''}, x, \lambda y_i + (1-\lambda)y_i^{'}) \not\subset C_i(x)$. This shows that $\lambda y_i + (1-\lambda)y_i^{'} \in P_i(x)$ and $P_i(x)$ is convex for all $x \in X$. By (iii) and Theorem 2.1 it follows that for each fixed $y_i \in X_i$, $P_i^{-}(y_i)$ is open. Indeed, if $x \in X \setminus P_i^{-}(y_i)$, then there exists a net $\{x^{\alpha}\}$ in $X \setminus P_i^{-}(y_i)$ such that $x^{\alpha} \to x$. Therefore, $x \in X$ and $F_i(T_i(x^{\alpha}), x^{\alpha}, y_i) \subset C_i(x^{\alpha})$. Let $z_i \in F_i(T_i(x), x, y_i)$. By (iii) and Theorem 2.1 that $x \multimap F_i(T_i(x), x, y_i)$ is $l.s.c.$ for each $y_i \in X_i$. By Theorem 2.2, there exists a net $\{z_i^{\alpha}\}$ in $F_i(T_i(x^{\alpha}), x^{\alpha}, y_i)$ such that $z_i^{\alpha} \to z_i$. Therefore $z_i^{\alpha} \in C_i(x^{\alpha})$. Since $C_i : X \multimap Z_i$ is an $u.s.c.$ multivalued map with closed values, it follows from Theorem 2.3 that C_i is a closed multivalued map. Therefore, $z_i \in C_i(x)$ and $F_i(T_i(x), x, y_i) \subset C_i(x)$. We saw that $x \in X$. Therefore, $x \in X \setminus P_i^{-}(y_i)$ and $X \setminus P_i^{-}(y_i)$ is closed for all $y_i \in X$. This shows that $P_i^{-}(y_i)$ is open for all $y_i \in X_i$. Let $G_i : X \multimap X_i$ be defined by

$$G_i(x) = \begin{cases} S_i(x) \cap P_i(x) & \text{if } x \in A_i, \\ S_i(x) & \text{if } x \notin A_i. \end{cases}$$

Then $G_i(x)$ is convex for all $x \in X$. By (i), $x_i \notin P_i(x)$ for each $x = (x_i)_{i \in I} \in X$. Hence, $x_i \notin G_i(x)$ for each $x = (x_i)_{i \in I} \in X$. It is easy to see that $G_i^{-}(y_i) = [S_i^{-}(y_i) \cap P_i^{-}(y_i)] \cup [(X \setminus A_i) \cap S_i^{-}(y_i)]$. Since $S_i^{-}(y_i)$ and $P_i^{-}(y_i)$ is open for all $y_i \in X_i$, $G_i^{-}(y_i)$ is open for all $y_i \in X_i$. By(v), for each $x \in X \setminus K$, there exist $j \in I$, $y_j \in M_j$ such that $x \in G_j^{-}(y_j)$. Then by Theorem 2.4 that there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $G_i(\bar{x}) = \emptyset$ for all $i \in I$. Since $S_i(x)$ is nonempty for all $x \in X$, $\bar{x} \in A_i$ and $S_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for all $i \in I$. Therefore, for all $i \in I$, $\bar{x}_i \in clS_i(\bar{x})$ and $F_i(t_i, \bar{x}, y_i) \subset C_i(\bar{x})$ for all $y_i \in S_i(\bar{x})$ and for all $t_i \in T_i(\bar{x})$. □

Remark 3.1 In Theorem 3.1, we don't assume any convexity assumption on T_i. The methods, conclusions and assumptions are different from the previous result of this type of problems (see [25, 26]).

THEOREM 3.2 *For each $i \in I$, suppose that*

(i) *for each $x = (x_i)_{i\in I} \in X$ and each $t_i \in D_i$, $F_i(t_i, x, x_i) \cap C_i(x) \neq \emptyset$;*
(ii) *for each $(t_i, x) \in D_i \times X$, $y_i \multimap F_i(t_i, x, y_i)$ is $C_i(x)$–quasiconvex-like;*
(iii) $T_i : X \multimap D_i$ *and* $F_i : D_i \times X \times X_i \multimap Z_i$ *are* u.s.c. *with nonempty compact values;*
(iv) $\mathrm{cl}S_i : X \multimap X_i$ *and* $C_i : X \multimap Z_i$ *are* u.s.c. *multivalued maps.* $S_i^-(y_i)$ *is open for all $y_i \in X_i$ and $S_i(x)$ is a nonempty convex set for all $x \in X$;*
(v) *there exist a nonempty compact subset $K \subset X$ and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exist $j \in I$, $y_j \in M_j \cap S_j(x)$ such that $F_j(t_j, x, y_j) \cap C_j(x) = \emptyset$ for all $t_j \in T_j(x)$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for all $i \in I$, $\bar{x}_i \in cl S_i(\bar{x})$ and for each $y_i \in S_i(\bar{x})$, there exist $t_i \in T_i(\bar{x})$ such that $F_i(t_i, \bar{x}, y_i) \bigcap C_i(\bar{x}) \neq \emptyset$.

Proof. Let $P_i : X \multimap X_i$ be defined by

$$P_i(x) = \{y_i \in X_i : F_i(t_i, x, y_i) \bigcap C_i(x) = \emptyset \text{ for all } t_i \in T_i(x)\}.$$

$P_i(x)$ is convex for each $x \in X$. Indeed, if there exist $\lambda_0 \in [0, 1]$, $x_0 \in X$, u_i, $u_i^{'} \in P_i(x_0)$ such that $\lambda_0 u_i + (1 - \lambda_0)u_i^{'} \notin P_i(x_o)$, then there exist $t_i^0 \in T_i(x_0)$ such that $F_i(t_i{}^0, x_0, \lambda_0 u_i + (1 - \lambda_0)u_i^{'}) \bigcap C_i(x_0) \neq \emptyset$. Let $z_i \in F_i(t_i{}^0, x_0, \lambda_0 u_i + (1 - \lambda_0)u_i^{'}) \bigcap C_i(x_0)$, then by (ii),
either

$$F_i(t_i{}^0, x_0, \lambda_0 u_i + (1 - \lambda_0)u_i{}^{'}) \subset F_i(t_i{}^0, x_0, u_i) - C_i(x_0),$$

or

$$F_i(t_i{}^0, x_0, \lambda_0 u_i + (1 - \lambda_0)u_i{}^{'}) \subset F_i(t_i^0, x_0, u_i{}^{'}) - C_i(x_0).$$

Therefore, either there exists $v_i \in F_i(t_i{}^0, x_0, u_i)$ such that $z_i \in v_i - C_i(x_0)$ or there exists $v_i{}^{'} \in F_i(t_i{}^0, x_0, u_i^{'})$ such that $z_i \in v_i{}^{'} - C_i(x_0)$. Hence either

$$v_i \in z_i + C_i(x_0) \subset C_i(x_0) + C_i(x_0) \subset C_i(x_0)$$

or

$$v_i{}^{'} \in z_i + C_i(x_0) \subset C_i(x_0).$$

This shows that either

$$v_i \in F_i(t_i{}^0, x_0, u_i) \cap C_i(x_0) \neq \emptyset$$

or

$$v_i{}' \in F_i(t_i{}^0, x_0, u_i{}') \cap C_i(x_0) \neq \emptyset.$$

But u_i, $u_i' \in P_i(x_o)$, $F_i(t_i, x_0, u_i) \cap C_i(x_0) = \emptyset$ and $F_i(t_i, x_0, u_i') \cap C_i(x_0) = \emptyset$ for all $t_i \in T_i(x_0)$. This leads to a contradiction. This shows that for all $x \in X$, $\lambda \in [0, 1]$ and all u_i, $u_i' \in P_i(x)$, $\lambda u_i + (1-\lambda)u_i{}' \in P_i(x)$ and $P_i(x)$ is convex for all $x \in X$. $P_i^-(y_i)$ is open for each $y_i \in X_i$. Indeed, if $x \in \overline{X \backslash P_i{}^-(y_i)}$, then there exists a net $\{x^\alpha\}_{\alpha \in \Lambda}$ in $X \setminus P_i{}^-(y_i)$ such that $x^\alpha \to x$. Therefore, $x_\alpha \in X$ and $F_i(T_i(x^\alpha), x^\alpha, y_i) \cap C_i(x^\alpha) \neq \emptyset$. Let $z_i{}^\alpha \in F_i(T_i(x^\alpha), x^\alpha, y_i) \cap C_i(x^\alpha)$. By (iii) and Theorem 2.1 that for each $y_i \in X_i$, $x \multimap F_i(T_i(x), x, y_i)$ is an u.s.c. multivalued map with compact values. Let $L = \{x_\alpha\}_{\alpha \in \Lambda} \cup \{x\}$. Then L is a compact set. By Theorem 2.3 that $F_i(T_i(L), L, y_i)$ is a compact set in Z_i. Therefore $\{z_i{}^\alpha\}$ has a subnet $\{z_i{}^{\alpha_\lambda}\}$ converges to $z_i \in F_i(T_i(L), L, y_i)$. Since for each $y_i \in X_i$, the multivalued map $x \multimap F_i(T_i(x), x, y_i)$ and C_i are u.s.c. with compact values, it follows from Theorem 2.3 that for each fixed $y_i \in X_i$, $x \multimap F_i(T_i(x), x, y_i)$ and C_i are closed. Therefore, $x \in X$ and $z_i \in F_i(T_i(x), x, y_i) \cap C_i(x) \neq \emptyset$. This shows that $X \setminus P_i^-(y_i)$ is closed for each $y_i \in X_i$. Hence $P_i{}^-(y_i)$ is open for each $y_i \in X_i$. Let $A_i = \{x \in X : x_i \in clS_i(x)\}$. Then by (iv), A_i is closed. Let $G_i : X \multimap X_i$ be defined by

$$G_i(x) = \begin{cases} S_i(x) \cap P_i(x) & \text{if } x \in A_i, \\ S_i(x) & \text{if } x \notin A_i. \end{cases}$$

Then $G_i(x)$ is convex for all $x \in X$. By (i), $x_i \notin P_i(x)$ for each $x = (x_i)_{i \in I} \in X$. Therefore, $x_i \notin G_i(x)$ for each $x = (x_i)_{i \in I} \in X$. It is easy to see that $G_i^-(y_i) = [S_i^-(y_i) \cap P_i^-(y_i)] \cup [(X \backslash A_i) \cap S_i^-(y_i)]$. Since $S_i{}^-(y_i)$ and $P_i{}^-(y_i)$ are open for all $y_i \in X_i$, $G_i{}^-(y_i)$ is open for all $y_i \in X_i$, by (v), for each $x \in X \backslash K$, there exists $j \in I$, $y_j \in M_j$ such that $x \in G_j^-(y_j)$. Then it follows from Theorem 2.4 that there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that for all $i \in I$, $\bar{x}_i \in clS_i(\bar{x})$, and for each $y_i \in S_i(\bar{x})$ there exists $t_i \in T_i(\bar{x})$ such that $F_i(t_i, \bar{x}, y_i) \cap C_i(\bar{x}) \neq \emptyset$. □

Following the same argument as Theorem 3.2, we can prove the following theorem.

THEOREM 3.3 *For each $i \in I$, suppose that*

(i) *for each $x = (x_i)_{i \in I} \in X$ and each $t_i \in D_i$, $F_i(t_i, x, x_i) \not\subset (-\text{int}C_i(x))$;*
(ii) *for each $(t_i, x) \in D_i \times X$, $y_i \multimap F_i(t_i, x, y_i)$ is $C_i(x)-$quasiconvex-like;*
(iii) *$T_i : X \multimap D_i$ and $F_i : D_i \times X \times X_i \multimap Z_i$ are u.s.c. multivalued maps with nonempty compact values;*
(iv) *$clS_i : X \multimap X_i$ and $C_i : X \multimap Z_i$ are u.s.c. multivalued maps, $S_i{}^-(y_i)$ is open for all $y_i \in X_i$ and $S_i(x)$ is a nonempty convex subset of X_i for all $x \in X$;*

(v) *there exist a nonempty compact subset* $K \subset X$ *and a nonempty compact convex subset* M_i *of* X_i *for all* $i \in I$ *such that for each* $x \in X \setminus K$ *there exist* $j \in I, y_j \in M_j \cap S_j(x)$ *such that* $F_j(t_j, x, y_j) \subset (-\mathrm{int}C_j(x))$ *for all* $t_j \in T_j(x)$.

Then there exists $\bar{x} = (\bar{x_i})_{i \in I} \in X$ *such that for all* $i \in I$, $\bar{x_i} \in cl S_i(\bar{x})$ *and for each* $y_i \in S_i(\bar{x})$ *there exists* $t_i \in T_i(\bar{x})$ *with* $F_i(t_i, \bar{x}, y_i) \not\subset (-\mathrm{int}C_i(\bar{x}))$.

Proof. Let $P_i : X \multimap X_i$ be defined by

$$P_i(x) = \{y_i \in X_i : F_i(t_i, x, y_i) \subset (-\mathrm{int}C_i(x)) \text{ for all } t_i \in T_i(x)\}.$$

Then $P_i(x)$ is convex for each $x \in X$. Indeed, if $u_i, u_i' \in P_i(x)$ and $\lambda \in [0, 1]$. Then for all $t_i \in T_i(x)$,

$$F_i(t_i, x, u_i) \subset -\mathrm{int}C_i(x) \text{ and } F_i(t_i, x, u_i') \subset (-\mathrm{int}C_i(x)).$$

For any $t_i \in T_i(x)$, by (ii) either

$$F_i(t_i, x, \lambda u_i + (1-\lambda)u_i') \subset F_i(t_i, x, u_i) - C_i(x) \subset (-\mathrm{int}C_i(x)) - C_i(x) \subset (-\mathrm{int}C_i(x))$$

or

$$F_i(t_i, x, \lambda u_i + (1-\lambda)u_i') \subset F_i(t_i, x, u_i') - C_i(x) \subset (-\mathrm{int}C_i(x)) - C_i(x) \subset (-\mathrm{int}C_i(x)).$$

Therefore $\lambda u_i + (1-\lambda)u_i' \in P_i(x)$ for all $\lambda \in [0, 1]$ and $P_i(x)$ is convex. Then following the similar argument as Theorem 3.2, we can prove Theorem 3.3. □

With the same argument as in Theorem 3.1, we can prove the following theorem.

THEOREM 3.4 *For each* $i \in$ I, *suppose that*

(i) *for each* $x = (x_i)_{i \in I} \in X$ *and each* $t_i \in D_i$, $F_i(t_i, x, x_i) \cap (-\mathrm{int}C_i(x)) = \emptyset$;
(ii) *for each* $(t_i, x) \in D_i \times X$, $y_i \multimap F_i(t_i, x, y_i)$ *is* $C_i(x) - quasiconvex$;
(iii) $T_i : X \multimap D_i$ *and* $F_i : D_i \times X \times X_i \multimap Z_i$ *are l.s.c. multivalued maps*;
(iv) $clS_i : X \multimap X_i$ *and* $W_i : X \multimap Z_i$ *are u.s.c. multivalued maps,* $S_i^-(y_i)$ *is open for all* $y_i \in X_i$ *and* $S_i(x)$ *is nonempty for all* $x \in X$;
(v) *there exist a nonempty compact subset* $K \subset X$ *and a nonempty compact convex subset* M_i *of* X_i *for all* $i \in I$ *such that for each* $x \in X \setminus K$, *there exist* $j \in I, y_j \in M_j \cap S_j(x)$ *and* $t_j \in T_j(x)$ *such that* $F_j(t_j, x, y_j) \cap (-\mathrm{int}C_j(x)) \neq \emptyset$.

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for all $i \in I$, $\bar{x}_i \in clS_i(\bar{x})$, $F_i(t_i, \bar{x}, y_i) \cap (-\text{int}C_i(\bar{x})) = \emptyset$ for all $y_i \in S_i(\bar{x})$, and for all $t_i \in T_i(\bar{x})$.

4. Applications to Fixed Point Theorems

As simple consequences of Theorems 3.1 and 3.3, we establish the existence theorems for any collection of multivalued mappings.

DEFINITION 4.1 Let $(E, \|\cdot\|_E)$ and $(V, \|\cdot\|_V)$ be normed spaces, X be a subset of E and Y be a subset of V. Let $F: X \multimap Y$ be a multivalued map, F is said to be nonexpansive if for all $x, y \in X$, $u \in F(x)$, there exists $w \in F(y)$, such that $\|u - w\|_V \leqslant \|x - y\|_E$.

PROPOSITION 4.1 *Let $(E, \|\cdot\|_E)$ and $(V, \|\cdot\|_V)$ be normed spaces, X be a subset of E, and Y be a subset of V. Let $F: X \multimap Y$ be a nonexpansive multivalued map. Then F is l.s.c.*

Proof. Let $x \in X$ and $\{x_n\}$ be any sequence in X converges to x. Suppose $u \in F(x)$. If $x_n = x$ for some $n \in N$, then we let $u_n = u$. Since $F: X \multimap Y$ is nonexpansive, if $x_m \neq x$ for $m \in N$, there exists $u_m \in F(x_m)$ such that $\|u - u_m\| \leqslant \|x - x_m\|$. Therefore, $\|u - u_n\| \leqslant \|x - x_n\|$ for all $n \in N$. Hence $u_n \to u$. Then by Theorem 2.2 that $F: X \multimap Y$ is l.s.c.

THEOREM 4.1 *Let I be any index set. For each $i \in I$, let E_i be a Banach space with a* Fréchet *differentiable norm. X_i be a closed convex subset of E_i, D_i be a nonempty subset of X_i and $X = \prod_{i\in I} X_i$. For each $i \in I$, suppose that*

(i) *$T_i : X \multimap D_i$ is a l.s.c. multivalued map with nonempty values; and*
(ii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for each $i \in I$ such that for each $x \in X \backslash K$, there exist $j \in I$, $y_j \in M_j$, such that*

$$\langle x_j - y_j, J_j(t_j - x_j)\rangle < 0$$

for some $t_j \in T_j(x)$, where J_j is the duality mapping of E_j.

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x}_i \in T_i(\bar{x})$ for all $i \in I$.

Proof. Let $F_i(t_i, x, y_i) = \{\langle x_j - y_j, J_j(t_j - x_j)\rangle\}$. Then by Theorem 2.5 that $(t_i, x, y_i) \to \langle x_j - y_j, J_j(t_j - x_j)\rangle$ is a continuous function. Let $Z_i = \mathbb{R}$, $C_i(x) = [0, \infty)$ and $S_i(x) = X$ for all $x \in X$. Then by Theorem 3.1 that

$\bar{x}=(\bar{x_i})_{i\in I}\in X$ such that for all $i\in I$, $\bar{x_i}\in X_i$ and $\langle \bar{x_i}-y_i, J_i(t_i-\bar{x_i})\rangle \geqslant 0$ for all $y_i\in X_i$ and all $t_i\in T_i(\bar{x})$. Let $t_i\in T_i(\bar{x})$ and $y_i=t_i$, then

$$\|\bar{x_i}-t_i\|^2=\langle t_i-\bar{x_i}, J_i(t_i-\bar{x_i})\rangle \leqslant 0$$

for all $i\in I$. Therefore, $\bar{x_i}=t_i\in T_i(\bar{x})$ for all $i\in I$.

Remark 4.1 If for each $i\in I$, X_i is compact, then condition (ii) of Theorem 4.1 is satisfied. Indeed, if X_i is compact, then we take $K=\Pi_{i\in I}X_i=X$. Therefore $X\backslash K=\emptyset$ and condition (ii) of theorem 4.1 is satisfied.

COROLLARY 4.1 *Let I be any index set. For each $i\in I$, let E_i be a Hilbert space, X_i be a closed convex subset of E_i, D_i be a nonempty subset of X_i and $X=\prod_{i\in I}X_i$. For each $i\in I$, suppose that*

(i) *$T_i : X \multimap D_i$ is a l.s.c. multivalued map with nonempty values; and*
(ii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for all $i\in I$ such that for each $x=(x_i)_{i\in I}\in X\backslash K$, there exist $j\in I$, $y_j\in M_j$ such that $\langle x_j-t_j, y_j-x_j\rangle<0$ for some $t_j\in T_j(x)$.*

Then there exists $\bar{x}=(\bar{x_i})_{i\in I}\in X$ such that $\bar{x_i}\in T_i(\bar{x})$ for all $i\in I$.

Proof. Let $S(E_i)=\{x_i\in E_i\colon\ \|x_i\|=1\}$, then $\lim_{t\to 0}(\|x_i+ty_i\|-\|x_i\|)/t=\lim_{t\to 0}\ (\|x_i+ty_i\|^2-\|x_i\|^2)/(t(\|x_i+ty_i\|+\|x_i\|))=\lim_{t\to 0}(2t < x_i, y_i > +t^2 \|y\|^2/)\ (t(\|x_i+ty_i\|+\|x_i\|))= < x_i, y_i >$ for all $x_i, y_i\in S(E_i)$. Therefore Hilbert space E_i has a *Fréchet* differentiable norm. Corollary 4.1 follows from Theorem 3.1. □

COROLLARY 4.2. *Let I, E_i be the same as Corollary 4.1. For each $i\in I$, let X_i be a closed bounded convex subset of E_i, D_i be a nonempty subset of X_i and $T_i : X=\prod_{i\in I}X_i \multimap D_i$ be a l.s.c. multivalued map. Then there exists $\bar{x}=(\bar{x_i})_{i\in I}\in X$ such that $\bar{x_i}\in T_i(\bar{x})$ for all $i\in I$.*

Proof. Since E_i is a reflexive Banach space and X_i is a closed bounded convex subset of E_i, X_i is weakly compact. Therefore $X=\prod_{i\in I}X_i$ is weakly compact. Then the conclusion of Corollary 4.2 follows from Theorem 4.1.

THEOREM 4.2 *In Theorem 4.1, if condition* (i) *is replaced by* (i′)*, then we have the same conclusion, where* (i′) *$T_i\colon X\multimap D_i$ is a nonexpansive multivalued map with nonempty values.*

Proof. Theorem 4.2 follows from Theorem 4.1 and Proposition 4.1.

If $T_i : X \multimap D_i$ is a single valued nonexpansive function, we have the following fixed point theorem.

COROLLARY 4.3 *Let I be any index set. For each $i \in I$, let E_i be a Hilbert space, X_i be a closed convex subset of E_i, D_i be a nonempty subset of X_i, $X = \prod_{i \in I} X_i$. and $T_i : X \to D_i$. For each $i \in I$, suppose that*

(i) *for all $x, \ y \in X$, $\|T_i(x) - T_i(y)\| \leqslant \|x - y\|$;*
(ii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X \backslash K$, there exist $j \in I$, $y_j \in M_j$ such that $\langle x_j - T_j(x), \ y_j - x_j \rangle < 0$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\bar{x}_i = T_i(\bar{x})$ for all $i \in I$.

COROLLARY 4.4. *Let E be a Hilbert space, X be a closed convex subset of E and D be a nonempty subset of X. Suppose that $T : X - \circ D$ satisfying the following conditions:*

(i) *T is a nonexpansive multivalued map;*
(ii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $x \in X \backslash K$, there exist $y \in M$, $u \in T(x)$ such that $\langle x - u, \ y - x \rangle < 0$.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.

COROLLARY 4.5 [6, 13, 16]. *Let E be a Hilbert space, X be a closed bounded convex subset of E, $T : X \to X$ be a nonexpansive map. Then there exists $\bar{x} \in X$ such that $\bar{x} = T(\bar{x})$.*

COROLLARY 4.6. *Let I be any index set. For each $i \in I$, let E_i be a Banach space with a* Fréchet *differentiable norm, X_i be a compact convex subset of E_i, D_i be a nonempty subset of X_i and $X = \prod_{i \in I} X_i$. For each $i \in I$, suppose that $T_i : X \multimap D_i$ is a l.s.c. or a nonexpansive multivalued map with nonempty values.*

Then these exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\bar{x}_i \in T_i(\bar{x}_i)$ for all $i \in I$.

Remark 4.2 Theorem 4.2 is different from the existing results of fixed point theorems of nonexpansive mappings even if I is a singleton and T_i is a single valued nonexpansive map. If I is a singleton, Theorem 4.1 is different from Theorem 1 [30].

THEOREM 4.3 *In Theorem* 4.1*, if condition* (i) *is replaced by* (i')*, then the conclusion of Theorem 4.1 is true, where*

(i') $T_i : X \multimap D_i$ *be a multivalued map with nonempty values and* $T_i^-(y_i)$ *is open for all* $y_i \in X_i$.

Proof. Since $T_i^-(y_i)$ is open for each $y_i \in X_i$, $T_i : X \multimap D_i$ is l.s.c. and the conclusion of Theorem 4.3 follows from Theorem 4.1. □

Remark 4.3 Theorem 4.3 is different from any generalization of Fan-Browder fixed point theorem [7]. In Theorem 4.3, $T_i(x)$ is not assumed to be a convex set for each $x \in X$ and $i \in I$.

THEOREM 4.4 *Let* I *be any index set. For each* $i \in I$*, let* E_i *be a Hilbert space,* X_i *be a closed convex subset of* E_i *and* $X = \Pi_{i\in I} X_i$*. For each* $i \in I$*, suppose that*

(i) $T_i : X \multimap D_i$ *is a multivalued map with nonempty values and* $T_i^-(y_i)$ *is open for all* $y_i \in X_i$*;*

(ii) *there exist a nonempty compact subset* K *of* X *and a nonempty compact convex subset* M_i *of* X_i *for all* $i \in I$ *such that for each* $x = (x_i)_{i\in I} \in X\backslash K$*, there exist* $j \in I$*,* $y_j \in M_j$ *such that* $\langle x_j - t_j, y_j - x_j \rangle < 0$ *for some* $t_j \in T_j(x)$.

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x}_i \in T_i(\bar{x})$ for all $i \in I$.

COROLLARY 4.7. *Let* I *be any index set. For each* $i \in I$*, let* E_i *be a Hilbert space,* X_i *be a closed bounded convex subset of* E_i*,* D_i *be a nonempty subset of* X_i*, and* $X = \prod_{i\in I} X_i$*. For each* $i \in I$*, suppose that* $T_i : X \multimap D_i$ *is a multivalued map with nonempty values and* $T_i^-(y_i)$ *is open for all* $y_i \in X_i$*. Then there exist* $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *such that* $\bar{x}_i \in T_i(\bar{x})$ *for all* $i \in I$.

Proof. Since X_i is weakly compact, $X = \prod_{i\in I} X_i$ is weakly compact and condition (ii) of Theorem 4.4 is satisfied. Then the conclusion of Corollary 4.4 follows from Theorem 4.6.

5. Existence Theorems of System of Generalized Vector Quasi-Mixed Variational-like Inequalities Problems

LEMMA 5.1. [11] *Let* W *and* Z *be Hausdorff t.v.s. and* $L(W, Z)$ *be the t.v.s. with the* σ*-topology. Then the linear mapping* $\langle \cdot, \cdot \rangle : L(W, Z) \times W \to Z$ *is continuous in* $L(W, Z) \times W$.

THEOREM 5.1. *Let* I, E_i, V_i, X_i, C_i, D_i *and* S_i *be the same as in section* 3*. For each* $i \in I$*, let* $L(E_i, Z_i)$ *be equipped with* σ*-topology,*

$T_i: X \multimap L(E_i, Z_i)$ be a l.s.c. multivalued map with nonempty values, $D_i \subseteq L(E_i, Z_i)$, let $q_i: X_i \times D_i \to L(E_i, Z_i)$, $\eta_i: X_i \times X_i \to X_i$ and $g_i: X \times X_i \to Z_i$ be continuous vector-valued functions. For each $i \in I$, suppose that the following conditions hold:

(i) *$S_i: X - \circ X_i$ is a multivalued map with nonempty convex values, $S_i^-(y_i)$ is open for all $y_i \in X_i$, and $clS_i: X - \circ X_i$ is u.s.c.;*
(ii) *$C_i: X - \circ X_i$ is u.s.c., $\eta_i(x_i, x_i) = 0$, $g_i(x, x_i) = 0$ for all $x = (x_i)_{i \in I} \in X$;*
(iii) *for each $(t_i, x) \in D_i \times X$, $y_i \to \langle q_i(x_i, t_i), \eta_i(y_i, x_i)\rangle + g_i(x, y_i)$ is $C_i(x)$–quasiconvex; and*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for each $i \in I$ such that for each $x = (x_i)_{i \in I} \in X \setminus K$ there exist $j \in I$ and $y_j \in M_j \cap S_j(x)$ such that $\langle q_j(x_j, t_j), \eta_j(y_j, x_j)\rangle + g_j(x, y_j) \notin C_j(x)$ for some $t_j \in T_j(x)$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that for each $i \in I, \bar{x}_i \in clS_i(\bar{x})$ and $\langle q_i(\bar{x}_i, t_i), \eta_i(y_i, \bar{x}_i)\rangle + g_i(\bar{x}, y_i) \in C_i(\bar{x})$ for all $y_i \in S_i(\bar{x})$ and all $t_i \in T_i(\bar{x})$.

Proof. Let $F_i(t_i, x, y_i) = \{\langle q_i(x_i, t_i), \eta_i(y_i, x_i)\rangle + g_i(x, y_i)\}$. Then the conclusion of Theorem 5.1 follows from Lemma 5.1 and Theorem 3.1. □

Remark. Condition (i) of Theorems 5.1 and Theorem 5.1 [25] are different. In Theorem 5.1 [25], S_i is assumed to be l.s.c. The conclusion between these two theorems are also different.

COROLLARY 5.1 *Let $I, L(E_i, Z_i), T_i, D_i, q_i, \eta_i$ and condition* (i) *be the same as Theorem* 5.1. *For each $i \in I$, let $\varphi_i: X_i \to Z_i$ be a continuous function. For each $i \in I$, suppose the following conditions hold:*

(ii) *$C_i: X \multimap Z_i$ is u.s.c., $\eta_i(x_i, x_i) = 0$ for all $x = (x_i)_{i \in I} \in X$;*
(iii) *for each $(t_i, x) \in D_i \times X, y_i \to \langle q_i(x_i, t_i), \eta_i(y_i, x_i)\rangle + \varphi_i(y_i)$ is $C_i(x)$–quasiconcave; and*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for each $i \in I$ such that for each $x = (x_i)_{i \in I} \in X \setminus K$, there exist $j \in I$ and $y_j \in M_j \cap S_j(x)$ such that $\langle q_j(x_j, t_j), \eta_j(y_j, x_j)\rangle + \varphi_j(y_j) - \varphi_j(x_j) \notin C_j(x)$ for some $t_j \in T_j(x)$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that for each $i \in I, \bar{x}_i \in clS_i(\bar{x})$ and $\langle q_i(\bar{x}_i, t_i), \eta_i(y_i, \bar{x}_i)\rangle + \varphi_i(y_i) - \varphi_i(\bar{x}_i) \in C_i(\bar{x})$ for all $y_i \in S_i(\bar{x})$ and all $t_i \in T_i(\bar{x})$.

Proof. Let $g_i(x, y_i) = \varphi_i(y_i) - \varphi_i(x_i)$ in theorem 5.1. □

COROLLARY 5.2. *For each $i \in I$, let $\varphi_i: X \times X_i \to Z_i$ be a continuous function and $S_i: X - \circ X_i$ be a multivalued map with nonempty convex values. For each $i \in I$, suppose that the following conditions hold:*

(i) *clS_i: $X \multimap X_i$ is u.s.c. and $S_i^-(y_i)$ is open for all $y_i \in X_i$;*
(ii) *for each $x \in X$, $y_i \to \varphi(x, y_i)$ is $C_i(x)$–quasuconvex;*
(iii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x = (x_i)_{i\in I} \in X\backslash K$, there exist $j \in I$, $y_j \in M_j \cap S_j(x)$ such that $\varphi_j(x^j, y_j) - \varphi_j(x^j, x_j) \notin C_j(x)$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for all $i \in I$, $\bar{x}_i \in clS_i(\bar{x})$, $\varphi_i(\bar{x}^i, y_i) - \varphi_j(\bar{x}^i, \bar{x}_i) \in C_i(\bar{x})$ for all $y_i \in S_i(\bar{x})$.

Proof. Let $l_i \equiv 0$ and $g_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x^i, x_i)$ in Theorem 5.1. □

Remark 5.1. If $S_i: X \multimap X_i$ is defined by $S_i(x) = X_i$ for all $x \in X$ then Corollary 5.2 is reduced to the Nash vector equilibrium theorems.

Applying Theorem 3.2 and following the same arguments as Theorem 5.1, we can prove the following theorem.

THEOREM 5.2. *For each $i \in I$, suppose that $T_i : X \multimap X_i$ is an u.s.c. multivalued map with nonempty values. Suppose conditions* (iii) *and* (iv) *of Theorem* 5.1 *are replaced by* (iv′)*, where*

(iv′) *there exist a nonempty compact subset $K \subset X$ and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X\backslash K$, there exist $j \in I$, $y_j \in M_j \cap S_j(x)$ such that $\langle q_j(x_j, t_j), \; \eta_j(y_j, x_j)\rangle + g_j(x, y_j) \notin C_j(x)$ for all $t_j \in T_j(x)$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, such that for all $i \in I$, $\bar{x}_i \in clS_i(\bar{x})$ and for each $y_i \in S_i(\bar{x})$, there exists $t_i \in T_i(\bar{x})$ such that $\langle q_i(\bar{x}, t_i), \; \eta_i(y_i, \bar{x}_i)\rangle + g_i(\bar{x}, y_i) \in C_i(\bar{x})$.

Applying Theorem 3.3 and following the same argument as Theorem 5.1, we have the following theorem.

THEOREM 5.3. *For each $i \in I$, suppose $T_i : X \multimap D_i$ is an u.s.c. multivalued map with nonempty values. For each $i \in I$, suppose that conditions* (iv) *of Theorem 5.1 are replaced by* (iv′)*, where*

(iv′) *there exist a nonempty compact subset $K \subset X$ and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X\backslash K$, there exist $j \in I$, $y_j \in M_j \cap S_j(x)$ such that $\langle q_j(x_j, t_j), \; \eta_j(y_j, x_j)\rangle + g_j(x, y_j) \in (-\operatorname{int} C_j(x))$ for all $t_j \in T_j(x)$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for each $i \in I$, $\bar{x}_i \in clS_i(\bar{x})$ and for each $y_i \in S_i(\bar{x})$ there exists $t_i \in T_i(\bar{x})$ such that $\langle q_i(\bar{x}_i, t_i), \; \eta_i(y_i, \bar{x}_i)\rangle + g_i(\bar{x}, y_i) \notin (-\operatorname{int} C_i(\bar{x}))$.

Applying Theorem 3.4 and following the same argument as Theorem 5.1, we have Theorem 5.4.

THEOREM 5.4. *For each* $i \in I$, *suppose that* $T_i : X -\circ D_i$ *is a l.s.c. multi-valued map with nonempty values. Suppose condition* (iv) *of Theorem 5.1 is replaced by* (iv′), *where*

(iv′) *there exist a nonempty compact subset* $K \subset X$ *and a nonempty compact convex subset* M_i *of* X_i *for all* $i \in I$ *such that for each* $x \in X \backslash K$, *there exist* $j \in I$, $y_j \in M_j \cap S_j(x)$ *such that* $\langle q_j(x_j, t_j),\ \eta_j(y_j, x_j)\rangle + g_j(x, y_j) \notin (-\text{int}C_j(x))$ *for some* $t_j \in T_j(x)$.

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for all $i \in I$, $\bar{x}_i \in cl S_i(\bar{x})$, $\langle q_i(\bar{x}_i, t_i),\ \eta_i(y_i, \bar{x}_i)\rangle + g_i(\bar{x}, y_i) \notin (-\text{int}\, C_i(\bar{x}))$ for all $y_i \in S_i(\bar{x})$ and all $t_i \in T_i(\bar{x})$.

Remark 5.2. With the same arguments as corollary 5.1, we can show that Theorems 5.2–5.4 can be applied to establish the Debreu vector equilibrium theorems and the Nash vector equilibrium theorems.

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